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GEOMETRIC CHARACTERIZATION OF ISOGONAL TRAJECTORIES ON A SURFACE.*

BY JOSEPH LIPKA.

§ 1. The family of ∞^2 isogonal trajectories of a simple system of ∞^1 curves in the *plane* has formed the subject of several investigations. Cesàro† first stated the remarkable theorem that for the ∞^1 curves, of a family of ∞^2 isogonals, which pass through a given point, P, of the plane, the centers of curvature lie on a straight line (called the Cesàro line), or, in other words, the circles of curvature have another point, P', in common. Sheffers, in three papers,‡ discussed the same family of curves; in particular, he studied certain reciprocal families of isogonals arising from the point-to-point transformation of the plane set up by the correspondence of P and P' of the Cesàro theorem; also certain point-line transformations set up by the correspondence of a point P and its associated Cesàro line.

Kasner§ studied, for euclidean space, another class of curves, viz. natural families of curves, the extremals connected with variation problems of the type $\int F ds =$ *minimum*, where F is any point function and ds is the element of arc in the space considered.‖ In the plane, he found that for such a family of ∞^2 curves, the same theorem as stated above for isogonals holds true. He then characterized a natural family by the additional property that for the ∞^1 curves of the family which pass through a given point of the plane, the two circles of curvature which there hyperosculate their respective curves are orthogonal. Now considering a family of ∞^2 curves as being composed of ∞^3 lineal elements, he showed that if in any isogonal family we rotate each lineal element about its initial point through a right angle, keeping its curvature unchanged, the new ∞^3 lineal elements form a natural family, thus characterizing a family of isogonals and completing the work of Cesàro and Sheffers.¶

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* Presented to the American Mathematical Society, January 1, 1913.

† Lezioni di Geometria intrinsica, 1896; or Vorlesungen über Natürliche Geometrie, 1901, p. 148.

‡ Leipziger Berichte, 1898, pp. 261–294; ibid., 1904, pp. 105–116. Math. Annalen, 1905, vol. 60, pp. 491–531.

§ Trans. Am. Math. Soc., 2d Series, vol. 11 (1909), pp. 201–219.

‖ Among such extremals, the family of ∞^2 trajectories arising in a conservative field of force for the same constant of energy, s probably the most familiar example.

¶ Isogonals in the plane have also been characterized in a different way by W. M. Smith, in his dissertation, "Simply Infinite Systems of Plane Curves," 1912.

In a recent paper,* the author gave the geometric characterization of natural families of curves in a general curved space of n-dimensions, and briefly indicated that, in the case where $n = 2$, i. e., on an ordinary surface, the family of isogonals possesses one of the properties also possessed by a natural family. It is the purpose of this paper to give the *complete geometric characterization of isogonal trajectories on a surface,* and to exhibit the relations existing between isogonal and natural families of curves on any surface.†

§ 2. If we take an isothermal system of curves as parameter curves on the surface, we can write the element of arc length in the form

$$ds^2 = \lambda(u, v)[du^2 + dv^2]. \tag{1}$$

With this system of parameter curves, the angle, Ω, which a curve passing out in the direction dv/du makes with the v-parameter curve, becomes simply‡

$$\tan \Omega = \frac{dv}{du} = v'. \tag{2}$$

We can therefore represent any simple system of ∞^1 curves on the surface by

$$v' = \tan \Omega(u, v). \tag{3}$$

If we turn each lineal element u, v, v' about the point u, v through a constant angle α, we get

$$v' = \tan(\Omega + \alpha) \tag{4}$$

as the equation of the ∞^1 isogonals which cut (3) at a constant angle α To get the complete family of ∞^2 isogonals, we need merely differentiate once more and eliminate α, thus getting

$$v'' = (\Omega_u + \Omega_v v')(1 + v'^2) \qquad (\textit{type I}) \tag{5}$$

as the differential equation of the complete set of isogonals of the simple system (3).

On the other hand, starting out with the problem of finding the extremals connected with a variation problem of the type

$$\int F(u, v)ds = \int F\sqrt{\lambda(1 + v'^2)}du = \int H du = \text{minimum} \tag{6}$$

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* Trans. Am. Math. Soc., 2d Series, vol. 13 (1912), pp. 77–95.

† As the previous paper contains only a statement (without proofs) of the theorems concerning natural families on an ordinary surface ($n = 2$), it has seemed best to embody here a fuller treatment of these curves also.

‡ Throughout the paper primes refer to total derivatives with respect to u; and literal subscripts to partial derivatives.

and applying the Euler condition for the vanishing of the first variation, viz.,

$$H_v - H_{v'u} - v'H_{v'v} - v''H_{v'v'} = 0 \tag{7}$$

we get

$$v'' = [(\log F\sqrt{\lambda})_v - (\log F\sqrt{\lambda})_u v'](1 + v'^2) \qquad (\textit{type } N) \tag{8}$$

as the differential equation of the natural family of curves.

We notice that (5) and (8) are merely special forms of a more general type

$$v'' = (\psi - \phi v')(1 + v'^2) \qquad (\textit{type } V) \tag{9}$$

which reduces to *type I* or to *type N* according to the restriction

$$\psi_v + \phi_u = 0 \tag{10$'$}$$

or

$$\psi_u - \phi_v = 0. \tag{10$''$}$$

Equation (9) represents ∞^2 curves, of which ∞^1 pass through any given point, one in each direction on the surface.

§ 3. Consider a point P on the surface. The geodesic curvature and center of geodesic curvature of a curve through P are respectively the curvature and center of curvature of the orthogonal projection of the curve on the tangent plane to the surface at P. We shall take the point P as origin of coördinates, the tangent plane as the xy plane and the coördinate axes as the tangent lines to the u and v parapeter curves. Applying the formulas for the radius and center of geodesic curvature

$$\frac{1}{\rho_g} = \frac{v'' - [(\log\sqrt{\lambda})_v - (\log\sqrt{\lambda})_u v'](1 + v'^2)}{\sqrt{\lambda}(1 + v'^2)^{\frac{3}{2}}} \tag{11}$$

and

$$\xi = -\frac{v'}{\sqrt{1 + v'^2}}\rho_g, \qquad \eta = \frac{1}{\sqrt{1 + v'^2}}\rho_g \tag{12}$$

to the curves of *type V*, we get

$$\frac{1}{\rho_g} = \frac{[\psi - (\log\sqrt{\lambda})_v] - [\phi - (\log\sqrt{\lambda})_u]v'}{\sqrt{\lambda}(1 + v'^2)}. \tag{13}$$

Now if we keep u, v fixed and allow v' to vary, the center of geodesic curvature will describe a certain locus; we get this by eliminating v' and ρ_g from equations (12) and (13), thus arriving at the *linear* equation

$$[\phi - (\log\sqrt{\lambda})_u]\xi + [\psi - (\log\sqrt{\lambda})_v]\eta = \sqrt{\lambda}. \tag{14}$$

Hence we have the theorem:

Property A. *The locus of the centers of geodesic curvature of the* ∞^1 *curves of the type* V (*and therefore of type* I *or* N) *which pass through a given point, is a straight line.*

Conversely, if we have any system of ∞^2 curves on a surface, say

$$v'' = \chi(u, v, v'), \tag{15}$$

for which the centers of geodesic curvature of the curves through any point lie on a straight line, say

$$\alpha(u, v)\xi + \beta(u, v)\eta = \sqrt{\lambda(u, v)}, \tag{16}$$

we find, by introducing the values of ξ, η, ρ_g from (11) and (12), that

$$v'' = \{[\beta + (\log \sqrt{\lambda})_v] - [\alpha + (\log \sqrt{\lambda})_u]v'\}(1 + v'^2) \tag{17}$$

is the differential equation of the ∞^2 curves. This is of the *type* V. *Property* A *therefore characterizes the curves of type* V.

§ 4. For each curve c of *type* V which passes through a point P, we may draw the curve g which osculates (has 3-point contact with) c, and which has constant geodesic curvature (that of c at P) throughout. We call g the osculating geodesic circle of c. The question arises: in how many directions through P do the curves g hyperosculate (have 4-point contact with) the corresponding c curve? To answer this we need simply apply the condition

$$\frac{d(\rho_g)}{ds} = 0 \tag{18}$$

to the form (13), and we get

$$\alpha v'^2 + \beta v' + \gamma = 0, \tag{19}$$

where

$$\alpha = 4\lambda^2(\phi\psi - \phi_v) + 2\lambda\lambda_{uv} - 3\lambda_u\lambda_v,$$

$$\gamma = 4\lambda^2(\phi\psi - \psi_u) + 2\lambda\lambda_{uv} - 3\lambda_u\lambda_v,$$

$$\beta = 4\lambda^2[(\psi_v - \phi_u) - (\psi^2 - \phi^2)] + 2\lambda(\lambda_{uu} - \lambda_{vv}) - 3({\lambda_u}^2 - {\lambda_v}^2).$$

Hence, as a consequence of property A, *the curves of type* V *are such that there are two directions through every point in which the osculating geodesic circles hyperosculate their corresponding curves.*

For the two directions given by (19) to be perpendicular, it is evidently both necessary and sufficient that we have $\alpha = \gamma$, i. e., $\phi_v - \psi_u = 0$, which is the condition (10″) that the curves of *type* V should form a natural family. Hence we have the theorem:

Property B. *For the curves of a natural family, the two directions through each point in which the osculating geodesic circles hyperosculate their corresponding curves, are orthogonal.*

But there is no such simple property which decides whether a system of *type V* is a system of isogonals.

§ 5. As was observed in § 1, we may, in the plane, start with any isogonal family, rotate each of its ∞^3 curvature elements about its initial point through a right angle, preserving its curvature, and thus get a new set of ∞^3 elements which form a natural family. The question arises whether a similar process will hold for any surface, replacing " curvature " by " geodesic curvature," of course.

For any curve of *type V* given by ϕ, ψ, in a direction v', we have from (13),

$$(20) \qquad \frac{1}{\rho_g} = \frac{[\psi - (\log \sqrt{\lambda})_v] - [\phi - (\log \sqrt{\lambda})_u]v'}{\sqrt{\lambda(1 + v'^2)}}.$$

For a curve of *type V* corresponding to $\bar{\phi}$, $\bar{\psi}$ and in a direction perpendicular to the first direction, i. e., in the direction $-(1/v')$, we have

$$(21) \qquad \frac{1}{\rho_g} = \frac{[\bar{\phi} - (\log \sqrt{\lambda})_u] + [\bar{\psi} - (\log \sqrt{\lambda})_v]v'}{\sqrt{\lambda(1 + v'^2)}};$$

for the equality of these geodesic curvatures, we must have

$$(22) \quad \bar{\phi} - (\log \sqrt{\lambda})_u = \psi - (\log \sqrt{\lambda})_v, \qquad \bar{\psi} - (\log \sqrt{\lambda})_v = -\phi + (\log \sqrt{\lambda})_u.$$

Now if the first system is isogonal, then $\psi_v = -\phi_u$, and if the second system is natural, then $\bar{\phi}_v = \bar{\psi}_u$; applying these relations to the quantities involved in (22) we get

$$(23) \qquad (\log \lambda)_{uu} + (\log \lambda)_{vv} = 0.$$

But f K denotes the Gaussian curvature of the surface, we have*

$$(24) \qquad K = -\frac{1}{2\lambda}[(\log \lambda)_{uu} + (\log \lambda)_{vv}].$$

Hence, *the only surfaces on which the above mentioned transformation is possible are the surfaces of zero Gaussian curvature, i. e., developable surfaces.*

We must therefore modify the above mentioned transformation. We may easily do so by not requiring the preservation of the geodesic curvature. Thus we may write (20) in the form

$$(25) \qquad \frac{1}{\rho_g} = \frac{\psi - \phi v'}{\sqrt{\lambda(1 + v'^2)}} - \frac{(\log \sqrt{\lambda})_v - (\log \sqrt{\lambda})_u v'}{\sqrt{\lambda(1 + v'^2)}}.$$

By comparison with (11) we see that the second term of the right member represents the geodesic curvature of that curve of the system $v'' = 0$ which passes out in the direction v'. Now the equation

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* Bianchi, Vorlesungen über Differentialgeometrie, p. 68.

$$v'' = 0 \tag{26}$$

is evidently the differential equation of the isogonals of a simple (∞^1) system of curves which form an isothermal set on the surface, i. e., corresponding to $\Omega =$ *constant.* We must also note that (26) is also a natural family, corresponding to $F = \lambda^{-\frac{1}{2}}$. In general, *the only families on a surface which are both isogonal and natural are the isogonals of an isothermal system,* for we must have two restrictions on ϕ, ψ, viz.,

$$\phi_v - \psi_u = 0 \quad \text{and} \quad \phi_u + \psi_v = 0,$$

i. e., Ω satisfies the Laplacian

$$\Omega_{uu} + \Omega_{vv} = 0,$$

hence the curves (3) form an isothermal set. We might thus expect that the isogonals of an isothermal system, in their dual role, would lead to the solution of our problem.

In fact, if we subtract the geodesic curvatures of the family ϕ, ψ and the family $v'' = 0$, we get

$$\frac{\psi - \phi v'}{\sqrt{\lambda(1 + v'^2)}} \tag{27}$$

and in order that this expression shall represent the curvature of the family $\bar{\phi}$, $\bar{\psi}$ in the perpendicular direction $-(1/v')$ as given by (21), we must have

$$\bar{\phi} - (\log \sqrt{\lambda})_u = \psi, \qquad \bar{\psi} - (\log \sqrt{\lambda})_v = -\phi. \tag{28}$$

Then if the ϕ, ψ family is isogonal, i. e., $\psi_v = -\phi_u$, it follows immediately that $\bar{\phi}_v = \bar{\psi}_u$, i. e., the $\bar{\phi}$, $\bar{\psi}$ family is natural.

Defining *corresponding geodesic curvature elements* on a surface as two elements which have the same initial point and the same direction, we may state

PROPERTY C. *If from the geodesic curvature of the ∞^3 geodesic curvature elements composing an isogonal family, we subtract the geodesic curvature of the corresponding ∞^3 elements of the isogonals of an arbitrary isothermal system, and then rotate each element through a right angle, the ∞^3 new elements will form a natural family.*[*]

We thus have a test for an isogonal family. *Given any ∞^2 curves on a surface, construct the isogonals of an arbitrary isothermal system. In a given direction through a given point there passes one geodesic curvature element of each family; construct a new element through this point and perpendicular to this direction, and whose geodesic curvature is equal to the difference of the*

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[*] Of course the direction of rotation is the same for each element.

geodesic curvatures of the corresponding elements of the two families. If the new set of ∞^3 *elements thus obtained possesses the characteristic properties A and B of a natural family, then the original family is a family of isogonals.*

It is of interest to note that if, by this construction, the new set forms the family of geodesics, i. e., corresponding to $F = \textit{constant}$, then the original family must have been the isogonals of an isothermal system; in other words, the isogonals of all isothermal systems on the surface are transformed into the family of geodesics.

The *analytic* curvature transformation which changes an isogonal family into a natural family is given by

$$(\Gamma)\qquad \begin{gathered} u = u_1, \quad v = v_1, \quad v' = -\frac{1}{v_1'}, \\ v'' = -\frac{v_1'' - [(\log\sqrt{\lambda})_v - (\log\sqrt{\lambda})_u v_1'][1 + v_1'^2]}{v_1'^3}. \end{gathered}$$

This transforms

$$(5)\qquad v'' = (\Omega_u + \Omega_v v')(1 + v'^2)$$

into

$$v'' = \{[\Omega_v + (\log\sqrt{\lambda})_v] - [\Omega_u + (\log\sqrt{\lambda})_u]v'\}(1 + v'^2)$$

which is of *type N* where

$$(29)\qquad (\log F)_v = \Omega_v, \quad (\log F)_u = \Omega_u, \quad \therefore\ F = e^{\Omega}.$$

We may easily verify that the transformation Γ is the analytical statement of the geometric transformation above described, and we thus have the following *corollary to Property C.*

The isogonal family corresponding to the point function $\Omega(u, v)$ *is, by the theorem stated in property C, transformed into the natural family corresponding to the exponential of the same point function.*

It may be easily verified that, except for developable surfaces, the operations Γ^2 and Γ^3 do not give isogonal or natural families, but that the operation Γ^4 does give the original isogonal family.

The reason for the simplicity of the transformation from isogonal to natural families for developable surfaces, lies in the fact that we may choose the straight line generators as our intermediary isothermal system, and these have for their isogonal trajectories the geodesics of the surface.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.